„Mutationem motus proportionalem esse vi motrici impressae“ **or: How to Understand Newton’s Second Law of Motion, After All.**

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Abstract

Historians of science do know that Newton’s second law of motion is not compatible with the *F = ma* which classical mechanics is based on. The true meaning of Newton’s law, however, is controversially discussed. The law’s tenor reads: *„Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur“*, in English: *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.* In this paper I provide an analysis which unveils a „Newtonian Constant“ of proportionality between the „motive force impressed“ and the „change in motion“ produced by that force. If we accept this constant with dimensions [L/T] derived from Newton’s teaching, we obtain the basis for an *authentic* „Newtonian mechanics“ valid in macrophysics as well as in microphysics that needs no modern improvement whatever.

„Mutationem motus proportionalem esse vi motrici impressae" or: How to understand Newton's second law of motion, after all.

I

Two recently published books offer their services as an aid for the reader who wants to understand Sir Isaac Newton's *Principia* of 1687: N. Guicciardini's „Reading the *Principia*"[1], and I.B. Cohen's „A Guide to Newton's *Principia*", an introduction to a new translation of Newton's *magnum opus* by I.B. Cohen and Anne Whitman from Newton's Latin into English[2]. But Guicciardini and Cohen confusingly differ substantially in their presentations of Newton's most elementary principle, *the concept of force,* which Newton introduces with his second law of motion, and, unfortunately, both of them fail to meet its true sense.

The second law of Newton's theory of force and motion mathematically connects the concept of „force" as *cause* with its *effect* on the motion of a body[3]. In Newton's Latin, the law in its main contents reads *„Mutationem motus proportionalem esse vi motrici impressae"*[4]. Cohen and Whitman render these words correctly into *„A change in motion is proportional to the motive force impressed"*. I.B. Cohen in his „Guide" points out that Newton here introduces a concept of „impulsive" force because this force produces *finite* velocities, respectively *finite* motions, respectively *finite changes* in the motion of a body[5]. Since Newton defines „motion" by the product „mass times velocity" (*Principia*, def. 2), in using the symbols „m" for mass and „v" for velocity we shall be allowed to symbolize Newton's term *„change in motion"* by $\Delta(mv)$ - vector notation omitted[6]. Newton's *„impressed motive force",* if symbolized by F_i, should then fulfill the proportion $F_i \propto \Delta(mv)$ or, if rendered into an equation, $F_i = \Delta(mv) \times C$, with C serving as constant of proportionality.

Obviously such an *impulsive* force *„vis motrix impressa"* F_i differs from the common view of Newton's second law to introduce the concept of a *continuous* force, $F_c = m(dv/dt) = d(mv)/dt = ma$ (with a = acceleration), which concept classical mechanics is based on. Most significantly, this classical textbook concept lacks the constant of proportionality C to *mathematically* connect the cause „force" with its proportional effect on motion.

Gucciardini, though he explicitly takes the Cohen-Whitman translation as a basis, without making any reference to Cohen's different presentation simply presupposes and maintains the „classical" view of the second law by implicitly alleging its consistency with Newton's words[7]. Thus he eludes a conflict between Newton's and the „classical" concept of force, of which Cohen, on the other hand, is well aware. Cohen attacks the matter frontally by explicitly alleging that Newton didn't need to distinguish between the „impulsive" and the „continuous" form of „force", nor had he to bother with constants of proportionality to arise from different concepts of „force", rather he *„avoided the problem of dimensionality because he was dealing with ratios rather than equations"*[8], and in general: *„because the Principia sets forth a dimensionless physics"*[9].

Alas! The famous *Principia*, the bible of classical mechanics, which Newton based on the art of measuring by the help of geometry[10], *„a dimensionless physics*" ? Is not the dimension of a physical magnitude *the geometric measure* of the magnitude? Is not the aim to *measure* physical magnitudes such as times, spaces, forces, velocities, motions, accelerations etc. the central concern and object of Newton's theory of motion? Didn't experimental philosophy in general start with Galileo's successful attempt to measure the constant acceleration of uniformly accelerated motion through the ratio of velocity and time, i.e. to *identify the dimension* [v/t = L/T²] of acceleration, *expressed and measured in units of space* [L] *and time* [T] ? And why, for Heaven's sake, does Cohen allege and believe that a theory of motion which deals with ratios and proportions instead of equations *„avoids the problem of dimensionality*" ? Is it not true that Galileo's and Newton's theory is a *quantitative* geometric theory of motion, i.e. a theory *of measurement of motion* in terms of times and spaces, even if presented not in equations? How could such a theory ever be mathematically consistent, had it not first solved the problem of measurement, equal measurement of equal magnitudes, different measurement of different, including the consideration of consistent constants of proportionality - all of which is the „problem of dimensionality" ? Should not a „dimensionless physics", then, be a contradiction in terms?

II

A careful mathematical research with respect to the measurement or the dimensions of Newton's concepts of „impulsive" and „continuous" force has never before been carried out (with one exception[11]), on reasons similar to those which lead Guicciardini and Cohen to their insufficient presentations of the second law. The reasons are that scholars often rely on the opinions of authorities and make use of unwarranted presuppositions in matters which seem too difficult for an independent investigation. If confronted with inconsistencies, they often resort to again unwarranted authoritarian statements. Thus an erroneous presentation of a principle as basic as Newton's second law of motion may continue through generations.

If one wants to investigate this matter profoundly, one will have to base the research on Newton's *method of first and ultimate ratios* which, in eleven *Lemmata,* is introduced in the *Principia*, book I section 1, as Newton's mathematical tool; and *of course* this method deals with measurement, i.e. - to spite Cohen - *with the problem of dimensionality of physical magnitudes.*

Lemma X concerns the concept of „force". The germ of it reads (according to the Cohen-Whitman translation): *„The spaces which a body describes when urged by any finite force ..... are at the very beginning of the motion in the squared ratio of the times."*[12].

This measure - or dimension - „space over square of time" [L/T²], connected to continually accelerated motion as the dimension of acceleration *a*, has already been mentioned above as Galileo's finding. Newton, however, doesn't speak of a constant *continuous* acceleration „space in squared ratio of the times" of a continuously accelerated motion, rather he confines the validity of the measure [L/T²] to *„the very beginning of the motion"*. This is due to the fact that in *Lemma X* he doesn't refer to a *continuous,* rather to a *finite* force, to quote Newton's Latin: *„Spatia quae corpus urgente quacunque vi finita describit....sunt, ipso motus initio, in duplicata ratione temporum*"[13]. "Spatia quae corpus urgente quacunque vi finita describit" - that is: "The spaces a body describes if urged by a f i n i t e f o r c e ".

The matter has to be a bit expanded since it concerns a main difference between Newton's authentic theory and classical mechanics. The latter knows only one „force", and this „force" is *always and exclusively* connected to *continuous acceleration*, and thus it is always a *continuous* force. This continuously accelerating force may also be called an „infinite“ force, in so far as it produces an *infinite increase* of the velocity *v*, measured through the ratio of velocity per time unit [L/T²], or of the quantity of motion (*mv*), accordingly measured by [mL/T²], i.e. the „acceleration“ *a* of a body *m*. The latter is the case with free fall, and with circular motion also, where *the direction* of motion is changed ad infinitum.

But Newton's theory knows *different* concepts of „forces“ with *different* effects on a body's state of rest or motion: A concept of a *finite* „impulsive force“, producing *finite* quantities of velocity or motion, or of changes of motion, is introduced in his work (in def. 4 and in the second law) under the name of „vis motrix impressa“, the *impressed motive force*. It is this *finite* impulsive „vis motrix impressa“ to which Newton refers in *Lemma X* as *„quacunque vis finita“* (i.e. *any finite force).* A different concept of *infinite* „continuous“ force, as but a *source* (see def. 4) of *continually emerging impressed forces* to generate continual changes in the motion of bodies, is present in his work as „vis centripeta“, the *centripetal force.*

The case will be more clarified by the following two diagrams. Let a body, urged by an infinitely or constantly accelerating force, start its motion in A. The measure [L/T²] of this acceleration will then be represented by the straight line AB to show that this measure in this case is n o t confined to *„the very beginning of the motion only“* (as Newton's term *„ipso motus initio“* should be rendered precisely), but *is valid at every stage of progress of this motion*, from its beginning to infinity (*figure 1*). Now, on the contrary, let the body start in A, urged by a *finite* impulsive force which produces a *finite* velocity of motion. In this case, the acceleration of the body will show a maximum at the very beginning of the motion, and will reduce to zero when the body reaches its un-accelerated, uniform straightlined motion, i.e. the *momentum* generated by the impulsive „impressed force“ (*figure 2*). This development of acceleration which is represented by the *straight line* AB in *fig. 1*, will be given in *fig. 2* by the *curved line* AC.

figure 1

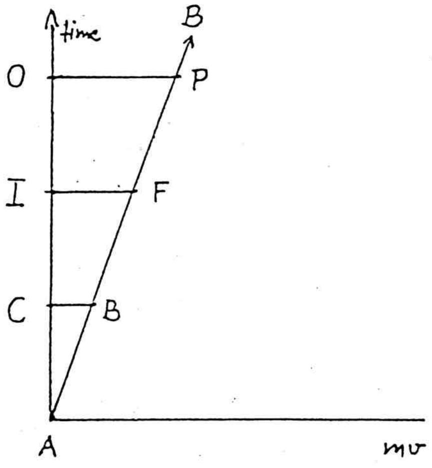
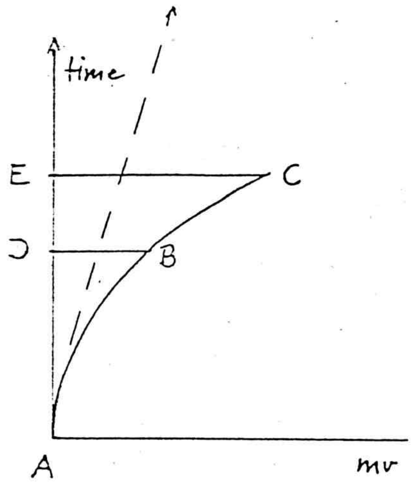


figure 2



In *fig. 1*, the velocities *v* respectively the motions or momenta *mv* produced in times AC, AI, AO, are given by CB, IF, OP. In *fig. 2*, the velocities *v* respectively the motions or momenta *mv* produced in times AD, AE, are given by DB, EC.

It should be noted that *fig. 1* is similar to Galileo's diagram representing the development of uniformly accelerated motion in his „Discorsi" of 1638[14] , while *fig. 2* is similar to the drawing Newton uses in the *Principia* to explain the action of a „finite" impulsive force according to *Lemma X*:

fig. 3 (Galileo)

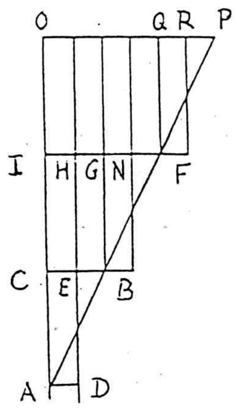


fig. 4 (Newton)

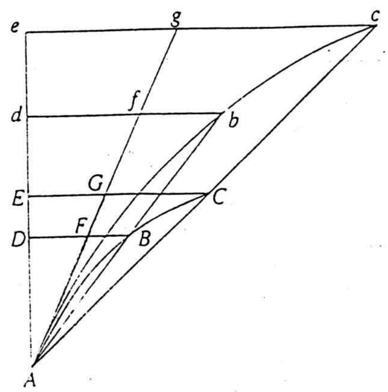


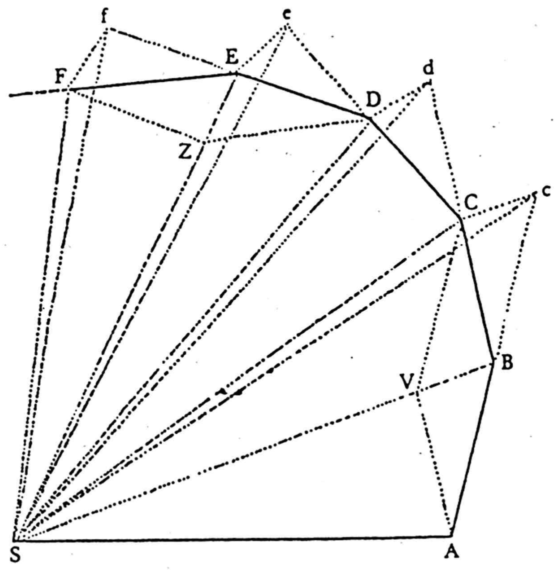
Fig. 3 is taken from the *Discorsi*, Third day, section *„De motu naturaliter accelerato,"* illustration to *Theorema II, Propositio II, Corollarium I.* As Galileo deals with the free fall of bodies, in his diagram point A, where the motion starts from, is *the top* of the figure, and OP is *the base* of the upside down triangle AOP.

Thus we can better understand *Lemma X* after we have freed ourselves from the general, but mistaken belief according to which Newton’s *Principia* should deal with always continuously accelerating „centripetal forces“ *only*. Quite the contrary, Newton’s def. 4 of „vis motrix impressa“ makes it clear that the concept of a *finite* „impressed motive force“ for Newton is *basic*, as it states that a (continuously acting) „vis centripeta“ is always *but a source* of such impressed forces. Says Newton, in the *Scholium* to follow def. 8*: „The causes which distinguish true motions from relative motions are the forces impressed upon bodies to generate motion. True motion is neither generated nor changed except by forces impressed upon the moving body itself.“* Motion is *neither generated nor changed* except by *forces impressed*." *Vis impressa*, the impressed *finite* force, is the basic concept of Newton's theory of motion. This can also be seen in Newton’s first law of motion, where we read that "every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by f o r c e s i m p r e s s e d" (my emphasis).

Keeping this in mind, we are ready to understand the reason of Newton’s demonstration of *Lemma X.* The reason is to show *that every* force, as a cause of motion, which basically is always a *finite* impulsive force F_i , *at the very beginning* of (the production of its proportional) motion DB, EC etc., but at the very beginning *only,* can be regarded as a *continuously acting* force F_c , and thus it can be computed according to Galileo’s space-over-time-squared-law of uniformly accelerated motion.

And what was this demonstration of Newton’s good for? It was good for the proof that the effect of an accelerating „vis centripeta“ on the motion of a body, which *infinite* force generates equal *finite* „impressed forces“ in equal times to produce equal velocities (of motion), or changes of velocities (or changes of motion) without end, can correctly be computed according to the space-over-time-squared-law, even though the measure or the dimension of the generated „impressed (finite) forces“ will fulfill this Galileian law „at the very beginning of the produced motions“ o n l y . This can now be understood as the essence of *figure 5* which Newton, immediately after the methodological introduction of sect. 1, presents as an illustration to *sect. 2 „To find centripetal forces“, Prop. 1 Theorem I* [15].

fig. 5



The diagram shows how from an *infinitely accelerating* „centripetal force“, directed to point S, there originate *finite* forces, which are *impressed on* the moving body at A, B, C, D, E, F, etc., in order to produce *finite changes* in the direction of motion which (by composition according to the laws of motion, *Corollary I*) deflect it from ABc, BCd, CDe, DEf, etc., to a path ABCDEF that *in the limit* describes a circular motion around the centre S.

III

One question remains for the careful reader: If impressed impulsive forces F_i to produce finite motions (or finite changes of motions, or *momenta*) can *only at the very beginning of the motion* (i.e. immediately when e.g. starting from rest) be measured according to the space-over-time-squared-law, how can such forces then be measured *in general*, say *without this confinement* to the very beginning of the motion? Newton clearly answers this question with his already quoted Second Law, to state that such forces are *proportional* to the produced motions or momenta, respectively to the produced finite changes in motion (including changes *in the direction of* motion). In Section 1 above we have found that the formula $F_i = \Delta(mv) \times C$ should correctly represent Newton’s second law. So, if we want to unveil the geometric measure, i.e. *the dimensions* of Newton’s F_i, we shall have to consider the dimensions of the product $\Delta(mv) \times C$. As the dimensions of the change of momentum $\Delta(mv)$ according to Newton’s definition of „motion“ (def. 2) are known to be [mL/T], our task will be to find the dimensions of C.

Now, if we should ask our guides Cohen and Guicciardini for help, we would feel somewhat disappointed. Cohen, as we have stated above, cannot see any problem here since he treats *finite* forces F_i (for which Cohen writes $F \propto d(mV)$) as if they were generally identical with (i.e. the same kind of force as) *infinite* forces F_c (for which Cohen writes $F \propto d(mV/dt)$). Moreover, Cohen ignores any factors of proportionality here, alleging that Newton, having conceived „a dimensionless physics", had not to bother with such things. Consequently Cohen falls back to the inacceptable position of simply identifying Newton's *finite* „impressed motive force" of the second law with Newton's „vis centripeta", and moreover with the *infinite* accelerating concept of „force equal (not proportional!) to mass times acceleration" of classical physics as well[16].

Surprisingly, Cohen somewhat later pretends to have understood the theory of proportions as Newton's most elementary mathematical tool. Especially as far as the application of proportion theory to relations of magnitudes of a different kind is concerned, Cohen, stating that Newton *„boldly"* allows *„that a quantity is proportional to a quantity of a wholly different kind"*[17], is well aware of Newton's use of „mixed proportions", .i.e. of the applicability of proportion theory to relations of heterogeneous magnitudes. And this is very clear and true the contents of Newton's *Scholium* following (not by chance) immediately to *Lemma X*, the *Scholium* giving some rules for the handling of relations between *„quantitates indeterminatae diversorum generum"*, i.e. variable magnitudes „of different kinds" (transl. Cohen-Whitman), as Newton does it in the preceding *Lemma X* (i.e. relations of such magnitudes as „force", „time", and „space"). However Cohen, in his „Guide", dedicates only five insignificant lines to that *Lemma*, and none at all to the said most important *Scholium*[18].

Turning now to our second guide Niccolò Guicciardini, we too shall find no answer to our question, since he, presupposing the „classical" $F = ma$ -concept as Newton's only concept of „force" in general, has no eyes for an impulsive finite „vis impressa" to produce finite proportional changes of motion. Actually, in his interpretation of *Lemma X*, Guicciardini mistakes Newton's finite force, ignoring the term „finite", for a *variably accelerating* force. Moreover, he raises our confusion to a higher level, as he steers clear of our question by simply alleging - in flagrant contradition even to Cohen - that Newton *was not at all able* to form a proportion between a „force" and a „change of motion", because his proportion theory *„does not allow the formation of a ratio between two heterogeneous magnitudes"*[19].

Alas, again. If Newton was not able to form *a ratio* between force and change of motion, how at all should he have been able to form even an *equation* (!) $F = ma$ between these *heterogeneous unequal magnitudes of a different kind* then? Should not the correctly understood *heterogeneity* of force (cause) and change of motion (effect) yield a striking argument *against* the idea to ascribe the *equation* $F = ma$ to Newton? Or, in other words: Is not the equation $F = ma$ an evident mathematical illustration of L e i b n i z ' s principle „causa a e q u a t effectum“, applied to a continuously mass-accelerating cause F? And, as far as Newton's use of proportion theory is concerned: Everybody who reads the *Principia*, the *Scholium* following *Lemma X*, will immediately see that Guicciardini's view contradicts not only Newton's clear words, but also Cohen's quite correct interpretation[20]. Moreover, as students of the history of proportion theory from Euclid via Tartaglia to Galileo, Torricelli, and John Wallis, do know, Guicciardini's view ignores and contradicts historical facts which are established by documentary evidence[21]. There is absolutely now doubt that Newton *of course* was in possession of the full Euclidean theory that included the theory of proportions of heterogeneous magnitudes (incommensurables). And it was exactly this knowledge which allowed him to state *that a quantity is proportional to a quantity of a wholly different kind* (to make use of Cohen's terms), as did already Galileo, when he formed the ratio „space over time squared“ (a ratio of quantities of a *very* different kind) to measure uniformly accelerated motions of e.g. falling bodies.

IV

Let us now concentrate on the problem of the constant of proportionality C that is as evidently required by Newton's second law as it is absent in the „classical“ mispresentation of this law. From Newton's *Lemma X* we know that a finite force F_i can in the limit be measured in the same way as an infinite force F_c. According to *Lemma X*, *Corollary 3*, the spaces [L] described by a body [m] under the influence of any force F_c, at the very beginning of the motion are as the product of the force F_c and the square of the time [i.e. T^2] :

$$L \propto F_c \times T^2 \qquad (1)$$

The measure of F_c then will be $F_c \propto L/T^2$ (2)

as it is stated in Newton's *Corollary 4* to *Lemma X*. Now, instead of this measure [L/T^2], I shall make use of the mathematically identical measure „velocity over time" [v/T]. Thus I obtain

$$F_c \propto v : T \quad (3)$$

which proportion is equivalent to the statement that the force F_c is to some hitherto unknown constant magnitude X, as the velocity v is to the time T :

$$F_c : X = v : T = \text{constant} \quad (4)$$

We should always be aware that this quaternary proportion is valid at the very beginning of the motion only. Now, to unveil the identity of X, we can make use of another such limited proportion which e.g. Roger Cotes introduced, in his preface to the *Principia's* second edition (1713). According to Cotes, it results from simple mathematical reasoning that the force, at the very beginning of the motion, (not only is proportional to the constant relation v/T, but also) is *proportional to the spaces described*. Writes Cotes: „*The rectilinear spaces described in a given time at the very beginning of the motion are proportional to the forces themselves*"[22], that is to say

$$F_c : L = \text{constant},$$

as well as (from (3)) $v : T = \text{constant}$

so that we obtain by composition

$$F_c : L = v : T \quad (5)$$

Remember now that $F_c = F_i$ at the very beginning of the motion. Consequently, L means an elementary finite length which is necessarily a constant element of space. However, since we are interested in the measure of the proportion of the force F_i to velocity v, or to motion mv, or to change of motion $\Delta(mv)$, *as it is stated in Newton's second law*, we may obtain by alternation[23]

$$F_i : \Delta(mv) = L : T = \text{constant} \ [L/T] \quad (6)$$

The measure, or the dimension, of the factor of proportionality to connect Newton's „vis motrix impressa" with its effect „mutatio motus" on the state of rest or motion of a body, now is unveiled to be given by [L/T], that is: *constant element of space* [L] *over constant element of time* [T] .

The true measure, or the dimension of Newton's finite „impressed force" F_i then will arise from

$$F_i \;\; [\text{mL/T} \times \text{L/T}] = \Delta(mv) \, [\text{mL/T}] \times C \, [\text{L/T}] \qquad (7)$$

One should be well aware that this measure of F_i *cannot be represented as a product* mL^2/T^2 of mL/T × L/T, because the first L/T stands for a *variable velocity*, whilst the second L/T stands for a *constant* relation of elements of „space" or length [L] and time [T]. It is clear that a product of *a variable* [L/T] and *a constant* [L/T] cannot be represented *as the square* [L^2/T^2] *of the variable or the constant.* Consequently, one would be misled if one would think of the above developed measure of „force" as a representation of the concept which Newton's philosophical antipode G.W. Leibniz left to physics under the name of „vis viva", the *living force,* today known as (kinetic) energy, with measure or dimensions [mL^2/T^2].

Nevertheless, it is interesting to see here how closely the Leibnizian concept of „living force" [mL^2/T^2] is related to Newton's „vis motrix impressa". As a matter of fact, Leibniz's concept results from ignoring the limitation of Newton's considerations „to the very beginning of the motion only", i.e. from taking the dimensions [L] and [T] of C not as constant elements of space and time, but rather as *variable* measures of *any* variable lengths and times, thus destroying the proportion of Newton's second law in favour of an *equality of cause and effect*[24], and *generalizing* eq. (5) at will, as a measure of *any* acting force at *any* variable time, and at *any* state of motion. In fact, if one does not think of a finite force F_i, as Newton did, the dimensions of which force *only at the very beginning of the motion* are given by the measure [mL/T^2], but of an *infinite constant* force F_c, the dimensions of which are *always* given by [mL/T^2], it can clearly be seen how the Leibnizian concept of kinetic energy [mL^2/T^2] results from eq. (5) *solved for* F_i, (which process is analogous to computing „kinetic energy" as *space integral of infinite force* according to the Leibnizian calculus). Note that in this case there appears *no constant of proportionality,* because its dimensions [L/T], erroneously *treated as variables*, are confounded with the dimensions of the variable „velocity" to form the squared space-over-time measure of this specific Leibnizian quantity of „living force“. And this may well have been one of the reasons why Newton accused those *„who confuse true quantities with their relations and common measures“* to *„corrupt mathematics and philosophy“* [25], and why he called Leibniz's *calculus „the analysis of the bunglers in mathematics“* [26].

In Newton's *authentic* theory of motion, as we have seen above, the „generalized“ measure of the *basic finite concept* of force F_i is not a „squared“, rather a „linear“ one, to be represented by

$$F_i = (mv) \times C \text{, or the equivalent } F_i = p \,\acute{}\, C \qquad (8)$$

with $p = mv = momentum$. Eq. (8) shows a close relationship between Newton's „vis motrix impressa“ and the equally „linear“ concept $E = p \times c$, or $E \propto p$ of the modern theory of propagation of light (in special relativity and quantum mechanics), with the constant of proportionality c to represent the absolute constant „vacuum velocity of light“ [L/T].

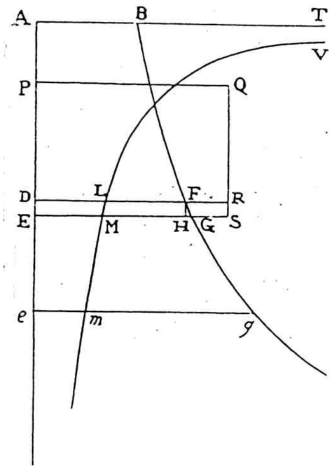
V

Another investigation for the true and complete dimensions of „force“ in Newton's authentic theory can be performed if one follows Newton's line of reasoning in the *Principia*, Book I, Section 8, proposition 41 concerning the determination of *„the orbits in which bodies revolve when acted upon by any centripetal forces“*. Extended analyses of this geometric proposition of Newton's are given by I.B. Cohen[27] and by N. Guicciardini[28]. Unfortunately, their common method *„in order to facilitate the understanding of this geometrical formula“* that Newton presents in prop. 41, is to *„betray* (!) *Newton and translate it into more familiar Leibnizian symbolic* [not geometric but algebraic] *terms“*, as Guicciardini puts it[29]; Cohen accordingly alleges that *„Newton's seemingly* (!) *geometric language enables us to translate his presentation rather directly into the more familiar* [algebraic] *algorithm of the Leibnizian calculus“*[30], and so does Guicciardini, as he states that Newton's geometry *„can be easily translated into (Leibnizian) calculus terms by substituting infinitesimal linelets for Newtonian moments (or Leibnizian differentials)“*[31] . In the following we shall see how this very substitution ignores the decisive difference between Newton's geometrical method and the

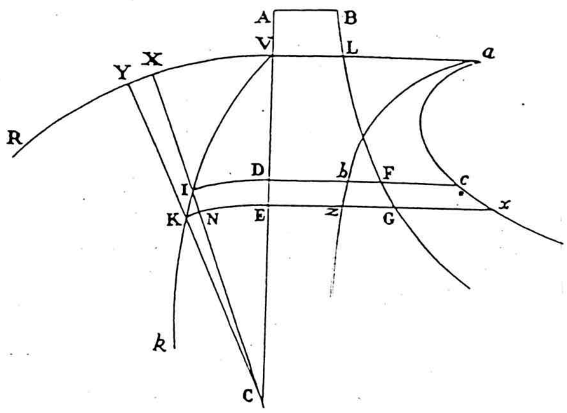
Leibnizian calculus, and thus corrupts Newton's fluxional method as well as his theory of motion by rendering an increment of a velocity (which is conceived as a *elementary, finite, constant quantity* in Newton's method, as will be shown) into a Leibnizian *variable* differential *ds/dt.*

Newton's prop. 41 draws on the preceding prop. 39. Both propositions are illustrated in the *Principia* by the following diagrams.

Prop. 39: (figure 6)



Prop. 41: (figure 7)



I shall concentrate on prop. 39 which concerns the case of *„a body ascending straight up or descending straight down"* from A, following the straight line ADEC, under the influence of a *centripetal force* of any (variable) kind. The task is put to find *„the velocity in any of its positions and the time in which the body will reach any place; and conversely"*. - As the body falls from A in the straight line ADEC, *„let there be always erected from the body's place E the perpendicular EG, proportional to the centripetal force in that place tending toward the centre C; and let BFG be the curved line which the point G continually traces out."* Now - says Newton - *„at the very beginning of the motion let EG coincide with the perpendicular AB; then the velocity of the body in any place E will be as the straight line whose square is equal to the curvilinear area ABGE. Q.E.I"*.[32] [Quod Est Inveniendum, i.e. what has to be found by demonstration]. "*In EG take EM inversely proportional to the straight line whose square is equal to the area ABGE, and let VLM be a curved line which the point M continually traces out and whose asymptote is the straight line AB produced; then the time in which the body in falling describes the line AE will be as the curvilinear area ABTVME. Q.E.I."*

In the subsequent paragraph to prove the proposition, Newton writes:

„In the straight line AE take a minimally small line DE of a given length, and let DLF be the location of the line EMG when the body was at D; then, if the centripetal force is such that the straight line whose square is equal to the area ABGE is as the velocity of the descending body, the area itself will be as the square of the velocity, that is, if V and V + I are written for the velocities at D and E, the area ABFD will be as V^2, and the area ABGE as $V^2 + 2VI + I^2$, and by separation [or dividendo] *the area DFGE will be as $2VI + I^2$, and thus DFGE/DE will be as $(2VI + I^2)/DE$, that is, if the first ratios of nascent quantities are taken, the length DF will be as the quantity 2VI/DE, and thus also as half of that quantity, or I ´ V/DE. But the time in which the body in falling describes the line-element DE is as that line-element directly and the velocity V inversely, and the force is as the increment I of the velocity directly and the time inversely, and thus - if the first ratios of nascent quantities are taken - as I ´ V/DE, that is, as the length DF. Therefore a force proportional to DF or EG makes the body descend with the velocity that is as the straight line whose square is equal to the area ABGE. Q.E.D."* [Quod Erat Demonstrandum, i.e. what had to be demonstrated]. *"Moreover, since the time in which any line-element DE of a given length is described is as the velocity inversely, and hence inversely as the straight line whose square is equal to the area ABFD, and since DL (and hence the nascent area DLME) is as the same straight line inversely, the time will be as the area DLME, and the sum of all the times will be as the sum of all the areas, that is (by lem. 4, corol.), the total time in which the line AE is described will be as the total area ATVME. Q.E.D."*

I shall now concentrate on the first „Q.E.D.", i.e. the proof for the task to find *the velocity* of the body in any place E. My aim is to make explicit *the geometric dimensions* of the quantities involved in units of „space" [L] and „time" [T], *in order to unveil the geometric dimensions of the centripetal force involved.*

Note that the centripetal force is always given through the lines AB, DF, EG etc. perpendicular to AC. Now, if (according to Newton) *„the first ratios of nascent quantities are taken*, *the length DF* [which represents a centripetal force F_c] *will be I × V/DE."* Since I and V mean velocities and DE means a length, the dimension of the variable centripetal force F_c represented by DF is given through I[L/T] × V[L/T] × 1/DE[1/L]. Taking into account that the velocity I according to Newton means an „*increment*“ of velocity, that is the velocity which is given through the rate of the „*minimally small line DE of a given length*“ over the again minimally small „*time in which the body in falling describes the line-element DE*“*,* and taking into account also that the minimally small „given length“ DE conceptually means an elementary *constant* quantity of length [L], the „*increment I of the velocity*“ will represent a *constant* quotient of an elementary unit of space over an elementary unit of time; I [L/T] = *constant.* From whence it follows that in Newton's above analyzed formula $F_c = I \times V/DE$ the only *variable quantities* are given through F_c and V. Consequently, we find that *the relation* of these variables, F_c /V = I/DE [L/T] $\times$ [1/L], must result in *a constant* with dimension [1/T]. And this result literally says that the quantities of centripetal force F_c and generated velocity V are *proportional*, connected by a constant *factor of proportionality* with dimension [1/T]. So we may interpret this result in harmony with Newton's def. 7 of the quantity (i.e. the geometric measure) of an accelerative centripetal force, according to which *the centripetal force F_c is proportional to the produced velocity V in a given (i.e. elementary constant) time T*; the „given time“ 1/T then means the dimension of the „constant of proportionality“ between this centripetal force and the proportional increment of velocity. Consequently we obtain for Newton's def. 7 and 8, with symbols F_c for „accelerating centripetal force“, *v* for „generated velocity“, and *m* for „mass“, and with constants of proportionality and their dimensions made explicit:

(def. 7) $F_c/v = \textit{constant}$ [1/T]

(def. 8) $mF_c = \textit{weight } G;\ G/m\,v = \textit{constant}$ [1/T] .

One should note, however, that *v* in both cases means an *increment* of velocity, i.e. that „first“ velocity which results from the quotient of a first given minimal length over a first given minimal time as a constant quantity.

Now, if we want to shift from F_c to F_i, in order to obtain *the generalized measure* of the *impressed* force F_i, , since Newton allows F_c as a measure of an *impressed* force F_i *at the very beginning of a motion only,* we must take into consideration that e.g. from some weight *G* [mL/T²] as a source, an *impressed force* as a measurable quantity will spring off (according to Newton's def. 4) if , and *only if* the weight (the body) *will actually have moved at least through a minimally small distance or length* [L]. Consequently, the already (in the past!)

„impressed“ force F_i which is proportional to the already performed (!) motion *mv* according to Newton’s second law, will be measured by the product of (weight *G* or) centripetal force F_c t i m e s L. And this measure $F_i = F_c \times L = mv \times [1/T] \times [L] = mv \times [L/T]$ unveils that the proportion F_i **:** *mv* (as stated in Newton’s second law) results in a constant factor with dimensions [L/T], which I have baptized the „Newtonian Constant“. Q.E.D.

This analysis shows and demonstrates how powerful dimensional analysis can be applied to Newton’s ratios and proportions, if one only proceeds carefully according to Newton’s clear words, and if one rejects Guicciardini’s proposal *to betray* (sic!) Newton by inconsiderately rendering his concepts into those of the Leibnizian calculus. As we can see now, the main difference between Newton’s and Leibniz’s concepts concerns the underlying structure of time and space. Since Newton holds a realist „quantized“ view which implies real elementary equal (and thus constant) particles of „space“ (length, [L]) and time, [T], his theory, when dealing with spaces and times *at the very beginning of motion*, or with an *increment of velocity* as well, must necessarily accept these elementary quantities *as natural constants* to constitute true geometric proportions between *variable* finite quantities such as „impressed force“ and „generated motion“ as soon as these quantities have appeared in reality. The *variable* quantities of spaces *really traversed* and times *really elapsed*, measured in relation to the absolute scales of space and time as represented by their constant elements [L] and [T], will then measure the *variable* velocity *v* of a really performed motion *mv*.

Leibniz, on the contrary, who conceived space and time not as real "absolute" entities, but only as structureless mathematical continua, consequently treats *every* appearing quantity of space (length) and time, and *every* increment of velocity *always as a variable*, even in the limit (Newton’s „ipso motus initio“), as it can be seen for instance in the case of the differentials *ds/dt* and *dv/dt.* Since he doesn’t accept any constant natural elements of space and time, he inevitably must *destroy* natural proportions based on such constants, in particular the proportion between force (cause) and motion (effect). In the case of how to measure a certain finite *impressed force* which has produced a certain *finite motion*, he must from $G \times L = [mv/T] \times [L]$, by taking L and T for variables *l* and *t*, proceed to a measure $mv \times l/t = mv^2$ $[mL^2/T^2]$ - the well-known „squared“ measure of „living force“ (the later "kinetic energy"). This is the "squared" concept which he, in the *vis-viva controversy*, from 1686 on promoted as his measure of force, against the „linear" concept of Newton to measure an impressed force *proportional to the produced motion* (according to the second law of motion)[33].

VI

The finding of a „Newtonian Constant" *C* [L/T][34] as a necessary part of Newton's second law of motion after all has settled the question which from 1686 on had nourished the *vis viva controversy* concerning the question of how to measure a finite impressed force. This result means certainly a decisive step in the full evolution of the geometric theory of motion of Galileo and Newton. It has been known for long that „classical mechanics", as it is taught in contemporary textbooks, differs very much from the teaching of these fathers of modern science. As we now can see, the main difference concerns the concept of impressed force. It concerns a constant of proportionality with dimensions „space over time" which had got lost in the 18th century when the synthetic-geometric theory of Galileo and Newton declined, as their followers rendered the language of rational mechanics into the algebraic-arithmetic terms of Descartes and Leibniz. „(Even) *In the hands of the early Newtonians, Newton's text moved from being a work in philosophy toward being the foundation of modern science"* (Margaret C. Jacob[35]). Our careful research now shows that the theory of analytical mechanics in the shape it has attained since the end of that century resulted from a violent and erroneous process of reducing geometric proportions to algebraic equations, erroneous and violent in so far as constants of proportionality were banned, transformed into variables, and cancelled *at will*.

Why do I call this process „erroneous", even though it brought forth so powerful a tool for mechanics and engineering as analytical mechanics certainly is? The answer is: because this process, through the above shown corruption of Newton's theory, created a law of motion *„F = ma"* which, as is well-known, proved deficient 100 years ago, and had to be replaced in modern physics by a better concept based on that very constant of proportionality which now has been revealed as an erroneously omitted, and for 300 centuries lost part of Newton's true theory. I mean, of course, *that absolute constant with dimensions „space over time"* which, under the name of „vacuum velocity of light" *c* [L/T], governs the most of modern physics. It should be noticed here that *„the view that a formal identity between mathematical relations* *betrays the identity of the physical entities involved harmonizes with the spirit of modern physics. Physical entities which satisfy identical formalisms have to be regarded as identical themselves"* (Max Jammer)[36]. Consequently, the Newtonian constant *c* [L/T] as a part of Newton's true theory will guarantee this theory the same exactness as we know it from the theories of modern physics thanks to the efficiency of the constant called „vacuum velocity of light" *c* [L/T] as a necessary part of a realist theory of motion.

Let me finally demonstrate that this constant „space over time" is already present in Galileo's teaching. Never has it been considered before *which set of units* Galileo used in his theory of motion. How - that is: *by means of which scale*, and *in which units* - did he measure lengths, how distances of fall? How - that means: *relative to which scale, and in which units* - did he measure times? Sometimes problems can be solved by asking the right questions. The answer to our question is that Galileo (as well as Newton afterwards) *made use of a set of units of „space"* [L] *and „time"* [T]. At the beginning of the most important part of his „Discorsi" of 1638, when he introduces the new theory of motion, Galileo draws two simple straight lines, one of them representing a scale of „space" (length, distance) *to measure variable „spaces" (lengths, distances) in units of space,* the other representing a scale of „time" *to measure variable times in units of time.*

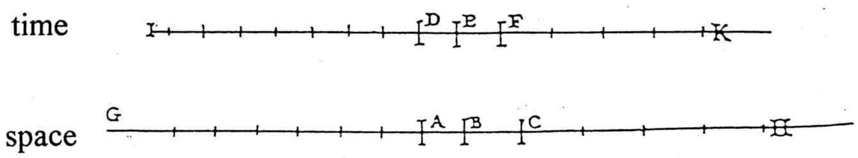


fig. 8 [37]

It is easy to understand that he who wants to measure the „spaces" and the „times" of bodies in motion will need two scales for this purpose. Galileo's scales contain, and are composed of, constant elementary parts or *units* of space [L] and of time [T], following one another *ad infinitum*. Thus Galileo's two innocent straight lines symbolize *geometrically* the metrics, i.e. the *quantization* of absolute space and time, *and the infinity of space as well as that of time* much in the way Giordano Bruno had taught it *literally*. This infinite scales of space and time, of course, in order to serve really as scales relative to which relative spaces (lengths, distances) and relative times can be measured, must necessarily be *graduated*, that is composed of finite constant elementary parts of „space“ [L] and „time“ [T]. And these elementary parts evidently stand to each other in a constant relation, which means that the elements of space and time are *proportional to each other*[38]. The constant proportion [L/T], then, can be called *the parameter which represents the metrics of the space-time frame of reference of motion*, as it lies behind the authentic theory of motion of Galileo as well as of Newton[39]. There is no doubt that this theory ever since required and tacitly included such a frame of reference of Euclidean shape, because a theory of motion without any such frame wouldn't make any sense. There is also no doubt that this frame is present in Galileo's drawing to explain the propagation of uniformly accelerated motion as in part already shown above, Section II *fig. 3*. The figure clearly reveals the always constant elements of space, BE, EC, FN, NG, GH, HI, PR, RQ, etc., and the always constant elements of time, AC, CI, IO etc. which together form *the space-time frame of reference* AOP wherein the accelerated motion starting in A takes place. If we now carefully analyze the proportions Galileo explains, we shall see that e.g. the rate of the increments of space traversed and the corresponding increments of time elapsed, *always results in a constant* [L/T] - the constant which, since I in 1983 found it in Newton's second law, I have termed „Newtonian Constant“. Guicciardini and others, who thought that Galileo only had formed series of homogeneous magnitudes such as $l_1:l_2:l_3:l_4$, and had compared this series with others, e.g. a series of times $t_1:t_2:t_3:t_4$, etc.[40] , should see that according to Euclid's definition book 5 def. 6, magnitudes *l* and *t* which have to each other the same relation (that means e.g.: l_1 has to l_2, l_2 has to l_3, l_3 has to l_4 etc. the same relation as t_1 has it to t_2, t_2 to t_3, t_3 to t_4 etc.), are termed *proportional*, i.e. that they result in a constant relation L/T. Since in Euclid's Greek „relation“ is „logos“, it is interesting to see that „proportional“ in Greek is „analogos“ which clearly indicates the difference between a *ratio* (logos) of *homogeneous* magnitudes, and a *proportion* (analogos) of *heterogeneous* magnitudes. The term „proportion“ should then above all indicate a constant relation between *heterogeneous* magnitudes.

As we can see now, the term „proportional“ in Galileo's and in Newton's theory, especially the *„proportionalem esse“* in Newton's second law, provides the constant space-time frame of reference and measurement of „spaces traversed“ and „times elapsed“ as variable values for the measurement of variable velocities and motions. No wonder, then, that the proportion-ality of these magnitudes to their generating forces, if made explicit in an equation, unveils the parameter L/T of an Euclidean space-time frame of reference.

VII

After all, the constant *C* [L/T] being a necessary part of Newton’s second law, represents nothing else but *the metrics of the Euclidean frame of reference of motion* which so many scholars in the past have thought to be not explicitly exposed (though implicitly presupposed) in Galileo’s and Newton’s theory. As this constant now stands clearly before our eyes, it stands there as a *parameter of the metrics of absolute space and absolute time* to serve as constant absolute scales for the measuring of variable and „relative“ spaces and times, „relative“ in so far as they are measured, and only can be measured, relative to these invariant scales of absolute space and of absolute time - a view which should agree with the contents of Newton’s extensive *Scholium* on space, time and motion to be read in the *Principia*, after def. 8. If accepted as a neccessary part of the second law of motion, this constant, by showing Newton’s concepts of absolute space and absolute time as indispensable mathematical constituents of the theory of real true (i.e. absolute) motion, will heal „classical“ mechanics from its main defect „instantaneousness“ (i.e. the unreasonable concept of motion to generate not in space and time, but instantaneously, i.e. without any elapse of time), thus giving back to Galileo and Newton the undefiled fame they deserve.

Alfred North Whitehead once said that Newton’s *Scholium* on space, time and motion, and Plato’s *Timaios*, contain the only two relevant cosmologies of western thought. But he didn’t realize that Newton’s philosophy of nature was heavily corrupted when, in the course of the 18th century, adherents of the relativist theory of motion of Descartes and Leibniz, by denying the existence of absolute space and absolute time (i.e. by denying the existence of natural scales for the measurement of variable times and spaces), and by equating the cause „force“ with its effect on motion, omitted the constant of proportionality, thus removed from the theory of motion together with the concepts of absolute space and absolute time the underlying absolute space-time frame of reference, and established a „classical mechanics“ which, under the false colours of Newtonianism, in fact rests on Leibniz’s relativism as to space and time, and on his concepts of „vis mortua“ ($F = ma$) and its space integral „vis viva“ ($E = mv^2$)[41].

Nothing in science can really be understood without the help of philosophy. In order to understand and reestablish the true authentic theory of Newton (and of Galileo), one must consider the philosophy of space and time on which it is founded, and follow an advice of - well - I. Bernard Cohen, who wrote some years ago: „*We must be careful lest we bind Newton's thinking in an intellectual strait-jacket that satisfies our own requirements at the expense of understanding his.*“[42].

One could not have said it better.

Postscript

Only after I had finished this paper I read Herman Erlichson's article on „Motive force and centripetal force in Newton's mechanics“, Am. J. Phys. 59 (1991), 842-9. There are some agreements, but also some disagreements to be noted as follows:

1) I agree with Erlichson's statement that Newton's basical concept of „motive force“ „originated in the consideration of collisional forces“ (p. 843), and that generally „Newton was thinking of the *finite* change of motion (proportional to the *finite* motive force)" (p. 843, my italics).

2) I disagree with Erlichson's view that „motive force“ should always act *instantaneously* (which is physically impossible, as we know from modern physics). Above I have tried to develop the mathematical description of impressed motive force and *change of motion generated in space and time*, as I find it in the *Principia*, especially in *Lemma* X.

3) I disagree with Erlichson as he identifies Newton's general concept of a (finite) „motive force“ with *Principia*, def. 8. In my view, def. 8 means explicitly that „the motive quantity *of centripetal force* is the measure *of this force* that is proportional to the motion which it generates in a given time“. (my italics). Erlichson, however, by inadmissibly generalizing this measure of *centripetal* force to mean plainly „force“ or „motive force“, confuses it with Newton's general and basic concept of a finite „vis motrix impressa“, the „impressed motive force“ which is defined in Newton's def. 4, and appears again as part of Newton's first and second law.

4) I agree with Erlichson's result that Newton's „continuous treatment which defines force at a point is based on the limit of the polygonal treatment. Newton's concept of force is always based on motive force" (p. 849). I want to add, however, that one should not speak of „motive force" here, and not refer to def. 8, but of *„impressed motive force"* which refers to the traditional Latin technical term „vis motrix impressa" (well-known to Galileo for instance) that is clearly defined and developed in Newton's *Principia* not in def. 8, but in def. 4, and in the second law. I have indicated above the far-reaching consequences which follow from this finding, supposed one is ready to depart from the un-Newtonian idea of instantaneousness and, taking into account the real development of motion *in space and time*, reveals carefully the dimensions of this concept of „impressed motive force".

Footnotes

1) Niccolò Guicciardini, Reading the *Principia*, Cambridge University Press, Cambridge 1999.

2) I. Bernard Cohen-Anne Whitman, The *Principia,* Mathematical Principles of Natural Philosophy, A New Translation, Preceded by A Guide to Newton's *Principia* by I. Bernard Cohen, University of California Press, Berkeley-Los Angeles-London, 1999.

3) Isaac Newton, Opera quae exstant omnia, Samuel Horsley ed., London 1779-1785, Vol.2 p. 12: *„Motus autem veros ex eorum causis, effectibus, & apparentibus differentiis colligere, & contra ex motibus, seu veris seu apparentibus, eorum causas & effectus, docebitur fusius in sequentibus. Hunc enim in finem tractatum sequentem composui."*

4) Isaac Newton Vol. 2 p. 14.

5) I.B. Cohen, A Guide to Newton's *Principia*, p. 110-3.

6) As Guicciardini does it (cf. p. 40), so do I omit the vector notation; it is (as a notation!) not necessary in this paper. Of course I am aware that to state *the vector quality* of „motion" *mv* is an important concern of Newton's second law.

7) Cf. Guicciardini p. 14, p. 40 („*Most notably force is often equated with acceleration*“).

8) Cohen, Guide, p. 56.

9) Cohen, Guide, p. 92.

10) See Newton’s „Preface to the Reader“ of May 8, 1686, Cohen-Whitman p. 381-2, where Newton praises geometry for being *„that part of universal mechanics which reduces the art of measuring to exact propositions and demonstrations“.*

11) See e.g. Ed Dellian, Inertia, the Innate Force of Matter, A Legacy from Newton to Modern Physics, in: P.B. Scheurer and G. Debrock (eds.), Newton’s Scientific and Philosophical Legacy, Kluwer Academic Publishers, Dordrecht 1988, p. 227-237.

12) Cohen-Whitman pp. 437-8.

13) Isaac Newton Vol. 2, p. 36.

14) Galileo Galilei, Discorsi e Dimostrazioni Matematiche intorno a Due Nuove Scienze attinenti alla Mecanica ed i Movimenti Locali, Giulio Einaudi (ed.), Torino 1990, p. 187.

15) Cohen-Whitman p. 444.

16) Cohen, Guide, p. 92, p.110-3, 116-7 (footnote 16 on p. 117).

17) Cohen, Guide, p. 312.

18) Cohen, Guide p. 130.

19) Guicciardini p. 126 (p. 125-8).

20) Cohen, Guide, p.312.

21) See Euclid, The Elements, Book V, definitions, as restored by Niccolò Tartaglia in his Italian „Euclide Megarense“ (1543); cf. Galileo Galilei, Discorsi, 5th day (on the theory of proportions); Evangelista Torricelli, Opere Geometriche, in: Opere scelte, L. Belloni ed., Torino 1975, p. 63; John Wallis, Mechanica sive De Motu Tractatus Geometricus, London 1670, prop. VII and Scholium. My point of view is confirmed by Stillman Drake, in: Galileo Galilei, Dialog über die beiden hauptsächlichsten Weltsysteme, R. Sexl and K. v. Meyenn Hrsg., Ergänzungen zu den Anmerkungen von Emil Strauss, Teubner, Stuttgart 1982, p. 578* footnote 33), and by R. Thiele, in: Geschichte der Analysis, Hans Niels Jahnke (Hrsg.), Spektrum Akademischer Verlag GmbH, Heidelberg-Berlin, 1999, p.18-9.

22) Roger Cotes, Editor’s Preface to the Second Edition of the *Principia*, in: Cohen-Whitman p. 385 (389).

23) Cf. Cohen, Guide, p. 313 on the method of alternation.

24) The result then is *causa aequat* (!) *effectum*, i.e. Leibniz’s „first axiom of mechanics“. Cf. Ernst Cassirer, Leibniz’ System, H. Olms, Hildesheim-New York 1980, p. 310-1.

25) Cohen-Whitman p. 414.

26) See Richard S. Westfall, Never at Rest, A Biography of Isaac Newton, Cambridge University Press, Cambridge 1980, p. 380.

27) Cohen's „Guide" p. 334-345 with references to important works of Derek T. Whiteside and Herman Ehrlichson on p. 335 fn. 18.

28) N. Guicciardini p. 218-222, with reference to Ehrlichson.

29) N. Guicciardini p. 221.

30) Cohen, Guide p. 335.

31) N. Guicciardini p. 59.

32) Cohen-Whitman p. 524.

33) Cf. Samuel Clarke, Der Briefwechsel mit G.W. Leibniz von 1715/1716, Ed Dellian ed., F. Meiner, Hamburg 1988, p. LXXV-LXXX.

34) First published in 1985 (Ed Dellian, Die Newtonische Konstante, Philos. Nat. 22 (1985) Vol.3, p. 400).

35) See Betty Jo Teeter Dobbs-Margaret C. Jacob, Newton and the Culture of Newtonianism, Humanity Books, New York 1998, p. 76.

36) Max Jammer, The Philosophy of Quantum Mechanics, New York 1974, p. 54.

37) The figure is taken from Galileo Galilei, Discorsi, as quoted in fn. 14.

38) Euclid, The Elements, Book V, def. 5.

39) Cohen too, as he claims that Newton in the *Principia* was „generally not concerned with units or with dimensionality" (Guide, p. 92), like many others fails to understand Newton's geometric theory of measurement in units of „absolute space" and „absolute time".

40) Cf. Guicciardini pp. 126-7 where he, by the way, mixes up the terms „ratio" and „proportio" at will.

41) Leibniz introduced these concepts in his „Specimen Dynamicum" of 1695 which was his answer to Newton's *Principia* of 1687.

42) I. Bernard Cohen, Newton's Second Law and the Concept of Force, in: The Annus Mirabilis of Sir Isaac Newton 1666-1966, R. Palter (ed.), Cambridge/Mass., 1970, p. 149.
